

Math 3235 Statistical Theory

3/9/2023

$$\underline{x} \quad \underline{y} \quad (\underline{x}, \underline{y}) = \sum_i x_i y_i$$

$$\|\underline{x}\|^2 = (\underline{x}, \underline{x})$$

$$A = A_{ij} \quad (A\underline{x})_i = \sum_j A_{ij} x_j$$

$$(\underline{x}, A\underline{x}) = \sum_i \sum_j x_i A_{ij} x_j$$

A is symmetric $A_{ij} = A_{ji} \quad \forall i, j$

$$A^T = A \quad (A^T)_{ij} = A_{ji}$$

A is positive definite if

$$(\underline{x}, A\underline{x}) > 0 \quad \forall \underline{x} \neq 0$$

if A is symmetric

There exist N $\underline{v}_k \quad k=1 \dots N$

$$A \underline{v}_k = \lambda_k \underline{v}_k$$

$$(\underline{v}_k, \underline{v}_j) = 0 \quad i \neq j$$

$$(\underline{v}_k, \underline{v}_k) = 1$$

If A is symmetric, There exists U such That

$$A = U^T D U$$

w. Th

D diagonal

U orthogonal

$$U^T U = Id$$

$$|\det U| = 1$$

$$\begin{aligned} \det A &= \det U^T \det D \det U \\ &= \det D \end{aligned}$$

A is positive definite \iff all eigenvalue are positive.

$$\int_{\mathbb{R}^n} f(\underline{x}) d\underline{x} = \int_{\mathbb{R}^n} f(U\underline{y}) |\det U| d\underline{y}$$

$$= \int_{\mathbb{R}^n} f(U\underline{y}) d\underline{y}$$

$\underline{x} = U\underline{y}$

A is positive definite matrix

$$f(\underline{x}) = \frac{1}{Z} e^{-\frac{1}{2}(\underline{x}, A\underline{x})}$$

Multivariate Normal distribution.

if there existed \underline{x} such that

$$(\underline{x}, A\underline{x}) < 0$$

$$(\lambda\underline{x}, A\lambda\underline{x}) = \lambda^2 (\underline{x}, A\underline{x}) \rightarrow -\infty$$

$$\frac{1}{e^{\lambda^2 (\underline{x}, A\underline{x})}} \rightarrow \infty \quad \lambda \rightarrow \infty$$

Non integrable.

Find Z

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(\underline{x}, A \underline{x})} d\underline{x} =$$

$$A = U^T D U$$

$$= \int_{\mathbb{R}^n} e^{-\frac{1}{2}(U \underline{x}, D U \underline{x})} d\underline{x}$$

$$(\underline{x}, U \underline{y}) = (U^T \underline{x}, \underline{y})$$

$$\underline{y} = U \underline{x}$$

$$= \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\underline{y}, D \underline{y})} d\underline{y}$$

$$(\underline{y}, D \underline{y}) = \sum_i y_i^2 \lambda_i$$

$$= \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_i y_i^2 \lambda_i} d\underline{y} =$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} e^{-\frac{1}{2} y_i^2 \lambda_i} dy_i =$$

$$z_i = \sqrt{\lambda_i} y_i$$

$$\int_{\mathbb{R}} e^{-\frac{1}{2} \lambda_i y_i^2} dy_i = \frac{1}{\sqrt{\lambda_i}} \int_{\mathbb{R}} e^{-\frac{1}{2} z_i^2} dz_i$$

$$= \sqrt{\frac{2\pi}{\lambda_i}}$$

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2} (\underline{x}, A \underline{x})} d\underline{x} = \prod_{i=1}^n \sqrt{\frac{2\pi}{\lambda_i}} =$$

$$= \frac{(2\pi)^{n/2}}{\left(\prod_i \lambda_i\right)^{1/2}} =$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

$$f(\underline{x}) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-\frac{1}{2} (\underline{x}, A \underline{x})}$$

$X_1 \dots X_n$

if joint p.d.f. is of the form

$$f(x) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(x-\mu, A(x-\mu))}$$

$\mu = (\mu_1, \dots, \mu_n)$ are real numbers.

Observe That

$$Y_i = X_i - \mu_i$$

Then the p.d.f of Y_i

is

$$f_Y(y) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(y, A y)}$$

$$E(Y_i) = \int_{\mathbb{R}^n} y_i f_Y(y) dy =$$

$$\underline{z} = \underline{y}$$

$$= \int_{\mathbb{R}^n} \underline{z}_i \underbrace{f_{\underline{y}}(\underline{z})}_{f_{\underline{z}}(\underline{z})} d\underline{z}$$

\Downarrow

$$\mathbb{E}(Y_i) = \mathbb{E}(Y_i)$$

\Downarrow

$$\mathbb{E}(Y_i) = 0.$$

$$\mathbb{E}(X_i) = \mathbb{E}(Y_i + \mu_i) = \mu_i$$

0

$$\mathbb{E}(Y_i Y_j) = \text{cov}(Y_i, Y_j)$$

$$\underline{Y} = U^T \underline{Z}$$

$$A = U^T D U$$

$$\mathbb{E}(Z_i Z_j)$$

$$f_{\underline{Z}}(\underline{z}) = f_{\underline{Y}}(U^T \underline{z}) =$$

$$= \frac{\sqrt{\det(A)}}{(2\pi)^{N/2}} e^{-\frac{1}{2}(\underline{z}, A \underline{z})}$$

$$= \frac{\sqrt{\det(A)}}{(2\pi)^{N/2}} e^{-\frac{1}{2}(\underline{z}, U A U^T \underline{z})}$$

$$A = U^T D U \quad \text{Id} \quad \text{Id}$$

$$U A U^T = U U^T D U U^T = D$$

$$\int_{\underline{z}} (\underline{z}) = \prod_i \frac{\sqrt{\lambda_i}}{2\pi} e^{-\frac{1}{2} \lambda_i z_i^2}$$

$$\mathbb{E}(z_i z_j) = \mathbb{E}(z_i) \mathbb{E}(z_j) = 0$$

$$\mathbb{E}(z_i^2) = \frac{1}{\lambda_i}$$

$$\mathbb{E}(z_i z_j) = \int_{\mathbb{R}^N} z_i z_j \prod_{k=1}^N \frac{\sqrt{\lambda_k}}{2\pi} e^{-\frac{1}{2} \lambda_k z_k^2} dz_k$$

$$= \frac{\sqrt{\lambda_i}}{2\pi} \int z_i e^{-\frac{1}{2} \lambda_i z_i^2} dz_i \cdot \frac{\sqrt{\lambda_j}}{2\pi} \int z_j e^{-\frac{1}{2} \lambda_j z_j^2} dz_j$$

$$\prod_{k \neq i, j} \frac{\sqrt{\lambda_k}}{2\pi} \int e^{-\frac{1}{2} \lambda_k z_k^2} dz_k$$

$$E(z_i^2) = \sqrt{\frac{\lambda_i}{2\pi}} \int_{\mathbb{R}} z_i^2 e^{-\frac{1}{2}\lambda_i z_i^2} dz_i$$

$$\prod_{k \neq i} \sqrt{\frac{\lambda_k}{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_k z_k^2} dz_k$$

$$t = \sqrt{\lambda_i} z_i$$

$$\sqrt{\frac{\lambda_i}{2\pi}} \int_{\mathbb{R}} z_i^2 e^{-\frac{1}{2}\lambda_i z_i^2} dz_i =$$

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\lambda_i} \int_{\mathbb{R}} \lambda_i z_i^2 e^{-\frac{1}{2}\lambda_i z_i^2} d(\sqrt{\lambda_i} z_i) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^2 e^{-\frac{1}{2}t^2} dt = 1$$

$$E(z_i z_j) = \lambda_i^{-1} \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$E(Y_i Y_j) = C_{ij}$$

$$Y = U^T Z \quad Y_i = \sum_j (U^T)_{ij} Z_j$$

$$C_{ij} = \sum_k \sum_e (U^T)_{ik} (U^T)_{je} E(Z_k Z_e)$$

$$E(Z_i Z_j) = D^{-1}$$

$$C = U^T D^{-1} U = A^{-1}$$

$$f(x) = \frac{\sqrt{\det A}}{(2\pi)^{N/2}} e^{-\frac{1}{2} (x, A x)}$$

$$\text{Cov}(X_i, X_j) = (A)^{-1}_{ij}$$

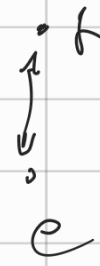
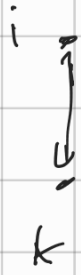
$$f(x) = \frac{1}{\sqrt{\det C} (2\pi)^{N/2}} e^{-\frac{1}{2} (x, C^{-1} x)}$$

$$\text{cov}(X_i, X_j) = C_{ij}$$

$$E(X_i X_j)$$

$$E(X_i X_j X_k) = 0$$

$$E(X_i, X_j, X_k, X_e)$$



$$C_{ik} \quad C_{ke}$$