Math 3235 Statistical Theory

3/9/2023

$\underline{x}$ $\underline{y}$ $(\underline{x}, \underline{y}) = \sum_i x_i y_i$

$\|\underline{x}\|^2 = (\underline{x}, \underline{x})$

$A = A_{ij}$ $(A\underline{x})_i = \sum_j A_{ij} x_j$

$(\underline{x}, A\underline{x}) = \sum_i \sum_j x_i A_{ij} x_j$

A is symmetric $A_{ij} = A_{ji}$ $\forall i, j$

$A^T = A$ $(A^T)_{ij} = A_{ji}$

A is positive definite if

$(\underline{x}, A\underline{x}) > 0$ $\forall \underline{x} \neq 0$

if A is symmetric

There exist N $\underline{v}_k$ $k = 1 \ldots N$

$$A \underline{v}_k = \lambda_k \underline{v}_k$$

$$(\underline{v}_k, \underline{v}_j) = 0 \quad i \neq j$$

$$(\underline{v}_k, \underline{v}_k) = 1$$

If A is symmetric, there exists U such that

$$A = U^T D U$$

with

D diagonal

U orthogonal

$$U^T U = Id$$

$$|\det U| = 1$$

$$\det A = \det U^T \det D \det U$$
$$= \det D$$

A is positive $\iff$ all eigenvalue are positive.

$$\int_{\mathbb{R}^n} f(\underline{x})\, d\underline{x} = \int_{\mathbb{R}^n} f(U\underline{y})\, |\det U|\, d\underline{y}$$

$$\underline{x} = U\underline{y} \qquad = \int_{\mathbb{R}^n} f(U\underline{y})\, d\underline{y}$$

A is positive definite matrix

$$f(\underline{x}) = \frac{1}{Z} e^{-\frac{1}{2}(\underline{x}, A\underline{x})}$$

Multivariate Normal distribution.

if there existed $\underline{x}$ such that

$$(\underline{x}, A\underline{x}) < 0$$

$$(\lambda\underline{x}, A\lambda\underline{x}) = \lambda^2 (\underline{x}, A\underline{x}) \to -\infty \quad \lambda \to \infty$$

$$e^{-\frac{1}{2}(\lambda\underline{x}, A\lambda\underline{x})} \to \infty \quad \lambda \to \infty$$

Non integrable.

Find Z

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\underline{x}, A\underline{x})} d\underline{x} =$$

$$A = U^T D U$$

$$= \int_{\mathbb{R}^N} e^{-\frac{1}{2}(U\underline{x}, DU\underline{x})} d\underline{x}$$

$$(\underline{x}, U\underline{y}) = (U^T\underline{x}, \underline{y})$$

$$\underline{y} = U\underline{x}$$

$$= \int_{\mathbb{R}^N} e^{-\frac{1}{2}(\underline{y}, D\underline{y})} d\underline{y}$$

$$(\underline{y}, D\underline{y}) = \sum_i y_i^2 \lambda_i$$

$$= \int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_i y_i^2 \lambda_i} d\underline{y} =$$

$$= \prod_{i=1}^{N} \int_{\mathbb{R}} e^{-\frac{1}{2} y_i^2 \lambda_i} \, dy_i =$$

$$z_i = \sqrt{\lambda_i}\, y_i$$

$$\int_{\mathbb{R}} e^{-\frac{1}{2} \lambda_i y_i^2} \, dy_i = \frac{1}{\sqrt{\lambda_i}} \int_{\mathbb{R}} e^{-\frac{1}{2} z_i^2} \, dz_i$$

$$= \sqrt{\frac{2\pi}{\lambda_i}}$$

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2} (\underline{x}, A \underline{x})} \, d\underline{x} = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} =$$

$$= \frac{(2\pi)^{N/2}}{\left(\prod_i \lambda_i\right)^{1/2}} =$$

$$= \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

$$f(\underline{x}) = \frac{\sqrt{\det A}}{(2\pi)^{N/2}} \, e^{-\frac{1}{2} (\underline{x}, A \underline{x})}$$

$X_1 \quad \dots \quad X_N$

if joint p.d.f. is of the form

$$f(x) = \frac{\sqrt{\det A}}{(2\pi)^{N/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu},\, A(\underline{x}-\underline{\mu}))}$$

$\mu = (\mu_1 \dots \mu_N)$ are real numbers.

Observe that

$$Y_i = X_i - \mu_i$$

Then the p.d.f of Y_i is

$$f_{\underline{Y}}(\underline{y}) = \frac{\sqrt{\det A}}{(2\pi)^{N/2}} e^{-(\underline{y},\, A\underline{y})}$$

$$\mathbb{E}(Y_i) = \int_{\mathbb{R}^N} y_i\, f_{\underline{Y}}(\underline{y})\, d\underline{y} =$$

$\underline{z} \to -\underline{y}$

$$= -\int_{\mathbb{R}^N} z_i \, f_{\underline{Y}}(\underline{z}) \, d\underline{z}$$

$\Downarrow$

$$\mathbb{E}(Y_i) = -\mathbb{E}(Y_i)$$

$\Downarrow$

$$\mathbb{E}(Y_i) = 0 .$$

$$\mathbb{E}(X_i) = \mathbb{E}(Y_i + \mu_i) = \mu_i$$

$$\mathbb{E}(Y_i \, Y_j) = \mathrm{cov}(Y_i, Y_j)$$

$$\underline{Y} = U^T \underline{Z} \qquad A = U^T D U$$

$$\mathbb{E}(Z_i \, Z_j)$$

$$f_{\underline{Z}}(\underline{z}) = f_Y(U^T \underline{z}) =$$

$$= \frac{\sqrt{\det(A)}}{(2\pi)^{N/2}} e^{-\frac{1}{2}\left(U^T_z,\ A U^T_z\right)}$$

$$= \frac{\sqrt{\det(A)}}{(2\pi)^{N/2}} e^{-\frac{1}{2}\left(\underline{z},\ U A U^T \underline{z}\right)} =$$

$$A = U^T D U$$

$$U A U^T = \cancel{U U^T}^{Id} D \cancel{U U^T}^{Id} = D$$

$$f_{\underline{z}}(\underline{z}) = \prod_i \frac{\sqrt{\lambda_i}}{2\pi} e^{-\frac{1}{2}\lambda_i z_i^2}$$

$$\mathbb{E}(z_i z_j) = \mathbb{E}(z_i)\,\mathbb{E}(z_j) = 0$$

$$\mathbb{E}(z_i^2) = \frac{1}{\lambda_i}$$

$$\mathbb{E}(z_i z_j) = \int_{\mathbb{R}^N} z_i z_j \prod_{k=1}^{N} \sqrt{\frac{\lambda_k}{2\pi}} e^{-\frac{1}{2}\lambda_k z_k^2} dz_k =$$

$$= \frac{\sqrt{\lambda_i}}{2\pi} \int z_i e^{-\frac{1}{2}\lambda_i z_i^2} dz_i \sqrt{\frac{\lambda_j}{2\pi}} \int z_j e^{-\frac{1}{2}\lambda_j z_j^2} dz_j$$

$$\prod_{k \neq i,j} \sqrt{\frac{\lambda_k}{2\pi}} \int e^{-\frac{1}{2}\lambda_k z_k^2} dz_k$$

$$\mathbb{E}(z_i^2) = \sqrt{\frac{\lambda_i}{2\pi}} \int_{\mathbb{R}} z_i^2 e^{-\frac{1}{2}\lambda_i z_i^2} dz_i \cdot \prod_{k \neq i} \sqrt{\frac{\lambda_k}{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_k z_k^2} dz_k$$

$$t = \sqrt{\lambda_i} z_i$$

$$\sqrt{\frac{\lambda_i}{2\pi}} \int_{\mathbb{R}} z_i^2 e^{-\frac{1}{2}\lambda_i z_i^2} dz_i =$$

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\lambda_i} \int_{\mathbb{R}} \lambda_i z_i^2 e^{-\frac{1}{2}\lambda_i z_i^2} d(\sqrt{\lambda_i} z_i) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^2 e^{-\frac{1}{2}t^2} dt = 1$$

$$\mathbb{E}(z_i z_j) = \lambda_i^{-1} \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\mathbb{E}(Y_i Y_j) = C_{ij}$$

$$Y = U^T Z \qquad Y_i = \sum_j (U^T)_{ij} Z_j$$

$$C_{ij} = \sum_k \sum_e (U^T)_{ik} (U^T)_{je} \mathbb{E}(Z_k Z_e)$$

$$\mathbb{E}(Z_i Z_j) = D^{-1}$$

$$C = U^T D^{-1} U = A^{-1}$$

$$f(x) = \frac{\sqrt{\det A}}{(2\pi)^{N/2}} e^{-\frac{1}{2}(\underline{x}, A \underline{x})}$$

$$\mathrm{Cov}(X_i, X_j) = (A)^{-1}_{ij}$$

$$f(x) = \frac{1}{\sqrt{\det C}\,(2\pi)^{N/2}} e^{-\frac{1}{2}(\underline{x}, C^{-1}\underline{x})}$$

$$\mathrm{cov}(X_i, X_j) = C_{ij}$$

$$\mathbb{E}(X_i X_j)$$

$$\mathbb{E}(X_i X_j X_k) = 0$$

$$\mathbb{E}(X_i \; X_j \; X_k \; X_e) =$$

$$\begin{tikzcd} i \arrow[d, leftrightarrow] & j \arrow[d, leftrightarrow] \\ k & e \end{tikzcd} \qquad C_{ik} \; C_{ke}$$